Review Packet of Algebra 2 Skills

Bainbridge High School

Helpful summer review for students **entering Precalculus or Honors Precalculus in the fall.**

Directions:

1. Study the review examples. You may print the packet or just view the packet online.
2. Do all the packet or just the sections that need extra review. Be sure to work on your own paper so that you have plenty of room to show the steps.
3. Check your answers using the provided answer key. Rework any problems you missed.
4. Bring completed work with you to your new class in the fall. Your new teacher can help you with any problems you don't understand.

Idea: Form a study group with your friends and work on it over the summer with fellow students.

Our thanks to Knoop's Regent Review for creating this excellent open source resource.

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Skill #1: Factoring and Solving Polynomial Equations

- Most common type on Algebra 2 Regents: GROUPING (4 terms)

Factoring Trinomials: You may have to take out a GCF first.

$x^2 - 6x - 16 = (x-8)(x+2)$

add multiply

$$2x^3 - 18x^2 + 28x = 2x(x^2 - 9x + 14)$$
$$= 2x(x-7)(x-2)$$

1. $4x^3 - 8x^2 - 32x$
2. $3x^3 + 21x^2 + 30x = 0$
3. $x^2 - 11x = -10$

Factoring by Grouping: You may need to reorder your expression first or factor out a GCF.

$2x^3 - x^2 + 8x - 4 = x^2(2x-1) + 4(2x-1)$ ⇒ Group the 1st two and last two terms & factor the GCF.

$= (2x-1)(x^2+4)$ ⇒ Take out the factor that is the same. The leftovers go in a set of parentheses together. If you can continue to factor, do so. If you can't, stop.

4. $x^3 - 2x^2 + 5x - 10$
5. $3y^4 + 9y^2 - 6y^3 - 18y$ ---- Take out a GCF first.
6. $4x^4 + 12x^3 + 6x^2 + 18x$
7. $3x + 7y + 21 + xy$

Factoring Trinomials Special Case: Leading Coefficient Is NOT A GCF

$2x^2 + 9x + 10$ First, multiply the leading coefficient by the last term: $2 \cdot 10 = 20$

- Think of two numbers whose product is 20 and whose sum is 9 (the middle term). The #s are 4 and 5.

$2x^2 + 4x + 5x + 10$ Split the middle term into two terms using those two numbers as coefficients.

$2x(x+2) + 5(x+2)$ Factor by grouping to finish.

$(2x+5)(x+2)$

8. $3x^2 + 10x + 8$
9. $8x^2 + 2x - 3$
10. $4x^2 - 3x - 10$

Skill #2: Polynomial Long Division

- Like regular long division, but with polynomials!

Long Division: Divide, Multiply, Subtract, Bring Down!

Write your answer in the form: $q(x) + \frac{r(x)}{g(x)}$, where $q(x)$ is the quotient, $r(x)$ is the remainder, and $g(x)$ is the divisor.

$$\frac{3x^4-12x+5}{x+1} \quad \Longrightarrow \quad \begin{array}{r} 3x^3 - 3x^2 + 3x - 15 \\ x+1\overline{)\,3x^4 + 0x^3 + 0x^2 - 12x + 5} \\ \underline{-3x^4 - 3x^3} \qquad\qquad\qquad\quad \\ -3x^3 + 0x^2 \qquad\qquad \\ \underline{3x^3 + 3x^2} \qquad\qquad \\ 3x^2 - 12x \quad\; \\ \underline{-3x^2 - 3x} \quad\; \\ -15x + 5 \\ \underline{15x + 15} \\ 20 \end{array}$$

This comes from $-(3x^4 + 3x^3)$. I prefer to distribute it out.

Same here. $-(-3x^3 - 3x^2)$

Same here. $-(3x^2 + 3x)$

Same here. $-(-15x - 15)$

So the final answer is: $3x^3 - 3x^2 + 3x - 15 + \frac{20}{x+1}$

1. $\frac{2x^3-5x^2-8x+15}{x-3}$

2. $\frac{x^4-8x^3+16x^2-19}{x-5}$

Skill #3: Remainder Theorem

- If f(a) = 0, then x-a is a factor. Keep reading to figure out what that means.

If $f(x) = 3x^4 - 12x + 5$ and $g(x) = x + 1$, find the remainder when $\frac{f(x)}{g(x)}$.

Set the denominator = to zero and solve: $x + 1 = 0 \Rightarrow x = -1$

Take your x value and plug it into the original: $f(-1) = 3(-1)^4 - 12(-1) + 5 = 3 + 12 + 5 = 20$

The value you obtain, 20, is the remainder. Since the remainder $\neq 0$, we know $x + 1$ is not a factor.

1. Determine the remainder when $p(x) = x^2 + 7x + 10$ is divided by $d(x) = x + 5$. Then complete the sentence below.

   $d(x)$ is/is not a factor of $p(x)$, since $p(\quad) =$______.
   CIRCLE ONE

2. Fill in the blanks: If $g(x)$ is a factor of $f(x)$ and $g(x) = x + 3$, then $f(\quad) =$ ________.

3. Fill in the blanks: If $m(x)$ is a factor of $g(x)$ and $m(x) = x - 1$, then $g(\quad) =$ ________.

4. Determine if $x - 2$ is a factor of $3x^3 - 4x^2 + x - 1$. Explain your answer.

5. Determine if $x + 5$ is a factor of $x^4 - 10x^2 - 375$. Explain your answer.

6. Given $f(x) = 4x^3 + 8x^2 - 3x + 27$, find $f(-3)$. What does your answer tell you?

7. Given $g(x) = 2x^3 - x + 8$, find $g(2)$. What does your answer tell you?

Skill #4: Undefined Fractions

- If the denominator of a fraction equals 0, then the fraction is undefined.

Determine the values that would make the fraction undefined.

$$f(x) = \frac{x^2 + 2x - 8}{x^2 - 3x - 10}$$

Set the denominator equal to zero and solve:

$$x^2 - 3x - 10 = 0$$
$$(x - 5)(x + 2) = 0$$
$$x = 5 \quad x = -2$$

So the domain is: $x \neq 5, x \neq -2$ (all real numbers except 5 and -2)

1. Find the values where the fraction is undefined:

$$\frac{x + 5}{x - 2}$$

2. What is the domain of:

$$f(x) = \frac{x^2 + 6x + 8}{x^2 - x - 6}$$

3. The fraction $f(x) = \frac{x-3}{x^2-4x-32}$ is undefined when x equals ____________________.

4. The fraction $f(x) = \frac{x+1}{x^2-4x-5}$ is undefined when x equals ____________________.

Skill #5: Simplifying Algebraic Fractions

- Apply fraction operations from middle school to algebraic fractions.

Given: $f(x)=\frac{x+2}{x-4}$ and $g(x)=\frac{3}{x-2}$

Multiply $f(x)\cdot g(x)=\frac{x+2}{x-4}\cdot\frac{3}{x-2}=\frac{3x+6}{(x-4)(x-2)}$

Divide $\frac{f(x)}{g(x)}=\frac{x+2}{x-4}\div\frac{3}{x-2}=\frac{x+2}{x-4}\cdot\frac{x-2}{3}=\frac{(x+2)(x-2)}{3(x-4)}=\frac{x^2-4}{3x-12}$

Add $f(x)+g(x)=\frac{x+2}{x-4}+\frac{3}{x-2}=\frac{(x+2)(x-2)}{(x-4)(x-2)}+\frac{3(x-4)}{(x-2)(x-4)}=\frac{x^2-4}{(x-4)(x-2)}+\frac{3x-12}{(x-4)(x-2)}=\frac{x^2+3x-16}{(x-4)(x-2)}$

Subtract $f(x)-g(x)=\frac{x+2}{x-4}-\frac{3}{x-2}=\frac{(x+2)(x-2)}{(x-4)(x-2)}-\frac{3(x-4)}{(x-2)(x-4)}=\frac{x^2-4}{(x-4)(x-2)}-\frac{3x-12}{(x-4)(x-2)}=\frac{x^2-3x+8}{(x-4)(x-2)}$

Simplify:

1. $\frac{x^2-3x-10}{2x^2+8x+8}$

2. $\frac{4-x^2}{x^2+2x-8}$

3. $\frac{3x^2+4x-4}{x^2+3x+2}$

4. $\frac{ax-a-c+cx}{acx-ac}$

Multiply or Divide:

5. $\frac{a+3}{a+2}\cdot\frac{a^2+3a+2}{a^2+4a+3}$

6. $\frac{m^2-n^2}{am+an}\div\frac{m-n}{m^2+n^2}$

Add or Subtract:

7. $\frac{x+3}{x-2}-\frac{x^2}{x^2-4}$

8. $2+\frac{3x}{2-x}$

Skill #6: Solving Rational Equations

- Get a common denominator then set the numerators equal. Check for values that make the original fractions undefined!

Undefined at $x = -2$ and $x = 3$

$$\frac{2}{x+2} - \frac{x+3}{x-3} = \frac{4x-1}{x^2-x-6}$$

$$\frac{2(x-3)}{(x+2)(x-3)} - \frac{(x+3)(x+2)}{(x+2)(x-3)} = \frac{4x-2}{(x+2)(x-3)}$$

$$\frac{2x-6}{(x+2)(x-3)} - \frac{x^2+5x+6}{(x+2)(x-3)} = \frac{4x-2}{(x+2)(x-3)}$$

$$2x - 6 - x^2 - 5x - 6 = 4x - 2$$

$$-x^2 - 3x - 12 = 4x - 2$$

$$-x^2 - 7x - 10 = 0$$

$$x^2 + 7x + 10 = 0$$

$$(x+5)(x+2) = 0$$

$$\boxed{x = -5} \quad \cancel{x = -2}$$

Reject since $x \neq -2$ since it would make the original fraction undefined

1. $\frac{x}{x^2-1} + \frac{2}{x+1} = \frac{1}{2x-2}$

2. $\frac{x-3}{x-7} - \frac{1}{x} = \frac{28}{x^2-7x}$

3. $\frac{1}{2x} - \frac{9}{x^2+6x} = \frac{2-x}{2x+12}$

Skill #7: Exponent Rules

- You know… all the rules from Algebra 1.

Multiplying: *Add Exponents* $\Rightarrow x^2 \cdot x^3 = x^5$ Dividing: *Subtract Exponents* $\Rightarrow \frac{x^6}{x^2} = x^4$

Zero Exponent: *Equals 1* $\Rightarrow x^0 = 1 \quad (3x)^0 = 1$ Power to A Power: *Multiply Exponents* $\Rightarrow (x^2)^3 = x^6$

Negative Exponents: *Become Fractional* $\Rightarrow x^{-2} = \frac{1}{x^2}$

Simplify:

1. $3x^3 \cdot x^9$
2. $(x^2y^3)(x^5y)$
3. $\frac{12x^6}{3x^3}$
4. $-\frac{10a^6b^2c}{5a^2bc}$
5. $(y^{-5})^{-3}$
6. $(3^a)^b$
7. $(2x^3)^4$
8. $(-m^2)^5$
9. $\left(\frac{3}{a^2}\right)^3$
10. $3x^0$
11. $(3x)^0$
12. x^{-3}
13. $6x^{-3}$
14. a^2b^{-3}
15. $\frac{x^{-2}}{x}$
16. $\frac{x^2y^{-3}}{x^{-3}y^{-2}}$
17. $\frac{a^4b^{-3}}{ab^{-2}}$
18. $\frac{3a^{-3}}{6b^{-2}}$
19. $\left(\frac{x^4y^{-2}}{x^5y^{-3}}\right)^{-2}$
20. $\frac{(2a^2b^4)^2}{2a^3b^{-5}}$

Skill #8: Rational Exponents ↔ Radical Form

- A rational (fractional) exponent can be converted into radical form:

$$x^{\frac{a}{b}} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$$

The ROOT is in the DENOMINATOR (bottom)!
Just like the root of a tree is in the ground (bottom)!

Rewrite $x^{\frac{2}{3}}$ as a radical: $\sqrt[3]{x^2}$ OR $\left(\sqrt[3]{x}\right)^2$ Rewrite $\sqrt[4]{a^3}$ as a rational exponent: $a^{\frac{3}{4}}$

Complete the Chart: The first one has been completed for you.

Radical	Fractional (Rational) Exponent	Simplified Form
$\sqrt{4}$	$4^{\frac{1}{2}}$	2
$\sqrt[3]{8}$		
	$16^{\frac{1}{2}}$	
	$16^{\frac{1}{4}}$	
$\left(\sqrt[3]{64}\right)^2$		
	$81^{\frac{3}{4}}$	
	$81^{-\frac{3}{4}}$	
$\frac{1}{\sqrt{25}}$		

Rewrite each radical expression as a power with a fractional exponent:

1. $\sqrt[3]{12}$
2. $\sqrt[5]{x^4}$
3. $\left(\sqrt[5]{2}\right)^3$
4. $x\sqrt{y^3}$

Rewrite with a radical sign instead of a fractional exponent:

5. $x^{\frac{1}{4}}$
6. $xy^{\frac{1}{4}}z^{\frac{3}{4}}$
7. $n^{\frac{2}{3}}$
8. $ab^{\frac{1}{2}}$

Rewrite with fractional exponents (if necessary) and simplify.

9. $x^{\frac{1}{5}} \cdot x^{\frac{2}{5}}$
10. $\frac{x^{\frac{7}{5}}}{x^{\frac{3}{4}}}$
11. $\sqrt{x} \cdot \sqrt[3]{x}$
12. $\frac{\sqrt[3]{x^2}}{\sqrt[6]{x}}$

Skill #9: Radical Equations

- Isolate the radical, then square both sides and solve. Check your answer! (Usually one answer gets rejected.)

$\sqrt{-x-1}+x=4x+5$

$\sqrt{-x-1}=3x+5$ Get the square root by itself.

$\left(\sqrt{-x-1}\right)^2=(3x+5)^2$ Square both sides.

$-x-1=(3x+5)(3x+5)$ Simplify.

$-x-1=9x^2+30x+25$ Simplify.

$9x^2+31x+26=0$ Set equal to zero.

$9x^2+13x+18x+26=0$ Factor (using grouping on this one).

$x(9x+13)+2(9x+13)=0$

$(9x+13)(x+2)=0$

$x=-\frac{13}{9}$ $\cancel{x=-2}$ Solve and check.

REJECT

Check:

$x=-\frac{13}{9}\Rightarrow$

$\sqrt{-\left(-\frac{13}{9}\right)-1}+\left(-\frac{13}{9}\right)=4\left(-\frac{13}{9}\right)+5$

$-0.\overline{7}=-0.\overline{7}$

So $x=-\frac{13}{9}$ is a solution.

$x=-2\Rightarrow$

$\sqrt{-(-2)-1}+(-2)=4(-2)+5$

$-2\neq-3$

So $x=-2$ is NOT a solution.

REJECT $x=-2$.

Solve each equation for x. Make sure you verify your solution by checking!

1. $x=1+\sqrt{x+5}$

2. $\sqrt{x^2-6x}=4$

3. $\sqrt{2x-7}-5=-x$

4. $3\sqrt{x-2}-2\sqrt{x+8}=0$

<u>Hint:</u> Move one square root over to the other side first (so that you have one square root on each side). Then square both sides.

Skill #10: Complex Number Operations

- You don't need to know this section. It's not real...... jk → $i^2 = -1$

Simplify: $4xi^2(-8xi - 2) = -16x^2i^3 - 8xi^2$ Distribute.

$= -16x^2(-i) - 8x(-1)$ Simplify: $i^3 = -i$ and $i^2 = -1$

$= 16x^2i + 8x$ Multiply.

Note: $i^0 = 1$ $i^1 = i$ $i^2 = -1$ $i^3 = -i$ $i^4 = 1$

CALCULATOR should be in $a + bi$ MODE!!

Simplify completely.

1. $ki(-k^2 + 3i)$
2. $2mi(i^2 + m)$
3. $(x + 3i)^2$
4. $(i - 5i)^2$

4. $2xi^3(5 + 2xi)$

6. $-7i(i^2 - 7i^2)^2$

7. $(1 - i)^3$

8. $(1 - xi)^3$

Skill #11: Quadratics with Complex Solutions

- When you use the quadratic formula, you might get a negative under the radical. This section shows you how to deal with that.

Solve for x: $2x^2 + 4x + 7 = 0$ Can't be factored, so use the quadratic formula.

$a = 2, b = 4, c = 7$

$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(7)}}{2(2)}$ Plug the values into the formula.

$x = \frac{-4 \pm \sqrt{-40}}{4}$ Simplify.

$x = \frac{-4 \pm \sqrt{-4}\sqrt{10}}{4}$ Break down the radical.

$x = \frac{-4 \pm 2i\sqrt{10}}{4}$ Simplify the radical.

$x = -\frac{4}{4} \pm \frac{2i\sqrt{10}}{4}$ Divide each term by the denominator.

$x = -1 \pm \frac{1}{2} i\sqrt{10}$ Simplify. This way of writing your answer is called $a + bi$ form.

Solve for x. Leave your answer in $a + bi$ form. (Note: This does not mean to only indicate the answer with the "+" sign. Technically, it means to leave your answer in $a \pm bi$ form.)

1. $8x^2 - 4x + 5 = 0$
2. $6x^2 + 2x = -5$
3. $2x^2 = -6x - 9$
4. $x^2 + 6x + 12 = 0$

Skill #12: Systems of Equations – 2x2

- Basic. These ones are not usually explicitly questioned on the Algebra 2 Regents, but they're necessary to understand how to do 3x3.

Solving by elimination: $-x + 5y = 8 \Rightarrow$ Multiply by 3 $\Rightarrow$ $3(-x + 5y = 8) \Rightarrow \cancel{-3x} + 15y = 24$

$3x + 7y = -2 \Rightarrow$ Leave the same $\Rightarrow$ $3x + 7y = -2 \Rightarrow \underline{\cancel{3x} + 7y = -2}$

Add them together: $22y = 22$

Solve: $y = 1$

Using $y = 1$, solve for x.

You can use either ORIGINAL equation! $-x + 5(1) = 8 \Rightarrow -x + 5 = 8 \Rightarrow -x = 3 \Rightarrow x = -3$

Final solution: $x = -3, y = 1$ so $(-3, 1)$ is the final solution.

Solve each system for x and y.

1. $5x - 2y = -19$

   $2x + 3y = 0$

2. $\frac{1}{2}x + \frac{2}{3}y = 1$

   $\frac{3}{4}x - \frac{1}{3}y = 2$

3. $5x - 3y = -1$

   $3x + 2y = 7$

4. $4x - 7y = 2$

   $3x - 3y = 6$

Skill #13: Systems of Equations – 3x3

- 3 equations, 3 unknowns – take it slow to avoid mistakes.

Solve by elimination: $x + 2y + z = 10$ Equation 1
$2x - y + 3z = -5$ Equation 2
$2x - 3y - 5z = 27$ Equation 3

Step 1: Choose two equations and eliminate one variable. I choose equations 2 & 3.
I will multiply equation 2 by -1 and eliminate the x's.

$$-\cancel{2x} + y - 3z = 5$$
$$\cancel{2x} - 3y - 5z = 27$$
$$-2y - 8z = 32$$

Step 2: Choose two more equations and eliminate the SAME variable as in Step 1. I choose equations 1 & 2. I will multiply equation 1 by -2.

$$-\cancel{2x} - 4y - 2z = -20$$
$$\cancel{2x} - y + 3z = -5$$
$$-5y + z = -25$$

$$-2y - \cancel{8z} = 32$$
$$-40y + \cancel{8z} = -200$$
$$-42y = -168$$
$$\boxed{y = 4}$$

Step 3: Take the two new equations and eliminate a variable. I will eliminate z's by multiplying $-5y + z = -25$ by 8. Once I get my solution for y, I will use one of these two equations again to find z.

$$-2y - 8z = 32$$
$$-2(4) - 8z = 32$$
$$-8 - 8z = 32$$
$$-8z = 40$$
$$\boxed{z = -5}$$

Step 4: Now that you have solutions for y and z, use one of the original equations to solve for x. I will use equation 1.

$$x + 2y + z = 10$$
$$x + 2(4) - 5 = 10$$
$$x + 3 = 10$$
$$\boxed{x = 7}$$

$\boxed{\text{Final Solution: } (7, 4, -5)}$

Solve each system for the three given variables.

1. $3x - 2y + 4z = 20$
   $-x + 5y + 12z = 73$
   $x + 3y - 2z = 1$

2. $x + 2y + 3z = 17$
$-4x + 2y - z = 24$
$3x - 6y - 8z = -67$

3. $4x + 2y - 2z = 10$
$2x + 8y + 4z = 32$
$30x + 12y - 4z = 24$

Skill #14: Systems of Equations – Circle/Quadratic and Line

- Where do a circle and a line intersect? Where do a parabola and a line intersect? Burning questions, I know.

Solve the following system of equations algebraically:

$(x-4)^2 + (y-3)^2 = 25$ (Circle)

$3x + 4y = 24$ (Line)

Step 1: Solve the line for x or y, whichever is easier to you. I'm going to solve for y.

$$4y = -3x + 24$$
$$y = -\frac{3}{4}x + 6$$

Step 2: Substitute your equation from Step 1 into the appropriate location in the circle equation.

$$(x-4)^2 + \left(-\frac{3}{4}x + 6 - 3\right)^2 = 25$$

Step 3: Simplify and solve.

$$(x-4)^2 + \left(-\frac{3}{4}x + 3\right)^2 = 25$$
$$(x-4)(x-4) + \left(-\frac{3}{4}x + 3\right)\left(-\frac{3}{4}x + 3\right) = 25$$
$$x^2 - 8x + 16 + \frac{9}{16}x^2 - 4.5x + 9 = 25$$
$$1.5625x^2 - 12.5x + 25 = 25$$
$$1.5625x^2 - 12.5x = 0$$
$$x^2 - 8x = 0 \quad \text{Divide by 1.5625.}$$
$$x(x-8) = 0$$
$$x = 0,\ x = 8$$

Solve for y: Substitute your x values into either of your original equations.

$x = 0$: $3(0) + 4y = 24$	$x = 8$: $3(8) + 4y = 24$
$4y = 24$	$24 + 4y = 24$
$y = 6$	$y = 0$

$\boxed{\text{Final Solutions: } (0,6) \text{ and } (8,0)}$ --- Note: Your final answers will NOT always have zero in them.

Solve each system algebraically.

1. $x^2 + y^2 = 100$
   $y - x = 2$

2. $2x + y = 15$
$(x - 2)^2 + (y - 1)^2 = 25$

3. $6x - 3 = y + x^2$
$7 = x + y$

4. $y + 4x = y - x^2 + 12$
$4 = 2x + y$

<u>Skill #15:</u> Systems of Equations – Random (Calculator Skill)

- Combo move: 2nd – Trace – Intersect

Tips and Tricks:

- How can you tell you might need to use your calculator?
  (1) They ask for where $f(x) = g(x)$ in a multiple choice question.
  (2) It's not an equation you know how to solve (like a log on one side and an absolute value on the other side).
  (3) It asks you to *round*. This means the intersection point isn't a whole number, which means it may not easily be solvable. (Make sure you round correctly!!)
- The solution is always the <u>x-value</u> of the intersection **unless they specifically ask for the point of intersection.** Sometimes the intersection points might be out of the standard view!
- You cannot use your calculator on a question in which you are told to solve <u>algebraically.</u> If told that you must solve algebraically, you could use this method to check. It would not be enough for full credit.

If $f(x) = \left(\frac{1}{3}\right)^x$ and $g(x) = |x + 5| - 2$, determine where $f(x) = g(x)$ to the *nearest tenth.*

<u>Solution:</u> Using your calculator: $y_1 = \left(\frac{1}{3}\right)^x$ and $y_2 = |x + 5| - 2$.
Graph and press 2nd – Trace – 5:Intersect
"First curve?" Arrow to the intersection point and hit enter.
"Second curve?" Arrow to the intersection pint and hit enter.
"Guess?" Hit enter.
Intersection: $x = -0.741552$ $y = 2.2584482$
Rounded to the *nearest tenth:* $x = -0.7$ -- Notice the y-value is not part of the solution.

Using your calculator, answer the following.

1. Determine the solution(s) to the system $f(x) = x^3 - 2x + 7$ and $g(x) = 2x + 5$.

2. Find the solution of $g(x) = h(x)$ if $g(x) = |2x + 3|$ and $h(x) = x^3$.

3. Find where $2^{x+3} = x + 5$.

4. Determine the <u>points</u> where $f(x) = k(x)$ if $f(x) = 3^x - 7$ and $k(x) = 3x^4 - 8$.

Skill #16: Average Rate of Change

- Just another phrase that means slope!

Slope Formula: Not on the reference sheet!!

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Given the table that shows inches of snowfall each day of a particular week, find the average rate of change between day 1 and day 4 and explain what it means in context.

Days	1	2	3	4	5	6	7
Snowfall	4.2	2	6.1	0.5	3	7.2	1

$$m = \frac{4.2 - 0.5}{1 - 4} = \frac{3.7}{-3} = -1.2\overline{3}$$

It means that the amount of snow that fell between day 1 and day 4 is decreasing on average by $1.2\overline{3}$ inches per day.

Using the slope formula, answer the following questions.

1. Which of the following functions has the largest average rate of change on the interval $[-3, 0]$?

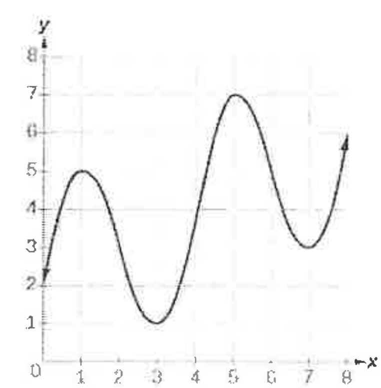
$$f(x) = |2x + 1| - 3$$

x	$g(x)$
-3	-4
-2	3
-1	-6
0	2
1	-8

2. The table shows the average diameter of a pupil in a person's eye as he or she grows older. Find the average rate of change from age 20 to age 80. Explain what this average rate of change means in context.

Age (years)	**Average Pupil Diameter (mm)**
20	4.7
30	4.3
40	3.9
50	3.5
60	3.1
70	2.7
80	2.3

3. Determine the average rate of change from $x = 2$ to $x = 5$.



Skill #17: Algebra of Functions

- Basic operations with polynomials as functions.

If $f(x) = 4x^2 + 2x - 1$ and $g(x) = 2x$, find:

- $f(x) + g(x) = 4x^2 + 2x - 1 + 2x = 4x^2 + 4x - 1$
- $g(x) - f(x) = 2x - (4x^2 + 2x - 1) = 2x - 4x^2 - 2x + 1 = -4x^2 + 1$
- $g(x) \cdot f(x) = 2x(4x^2 + 2x - 1) = 8x^3 + 4x^2 - 2x$
- $\frac{f(x)}{g(x)} = \frac{4x^2+2x-1}{2x} = \frac{4x^2}{2x} + \frac{2x}{2x} - \frac{1}{2x} = 2x + 1 - 0.5x^{-1}$

Simplify each of the following.

1. If $f(x) = 3x - 2$ and $g(x) = 5x + 7$, find $2f(x) + g(x)$.

2. If $f(x) = x^2 + 7x + 10$ and $g(x) = x^3$, find $\frac{f(x)}{g(x)}$.

3. If $f(x) = x + 1$ and $g(x) = 2x - 3$, find:
   a. $f(x) \cdot g(x)$

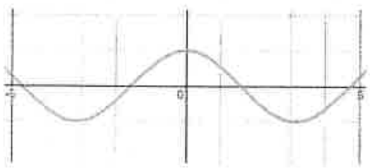
   b. $2[f(x) + 1]^2 - 3$

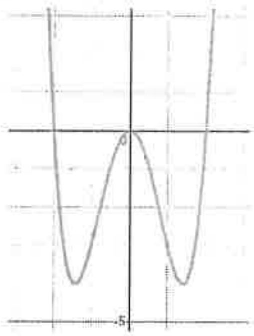
   c. $[g(x)]^2 + 5$

Skill #18: Special Functions – Even/Odd, Inverses

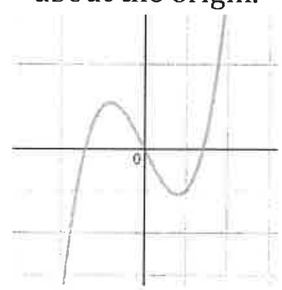
- Even Functions: Symmetric about the y-axis.
- Odd Functions: Symmetric about the origin.
- Inverse Functions ($f^{-1}(x)$): Switch x and y, then solve for y.

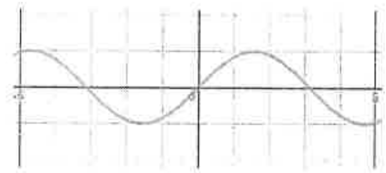
<u>Even Functions</u>: These functions have a line of symmetry on the y-axis.





<u>Odd Functions:</u> These functions are symmetric about the origin.





Algebraically: Classify each function as even, odd, or neither.

If even: $f(-x) = f(x)$
$f(-x)$ is the SAME as the original.

If odd: $f(-x) = -f(x)$
$f(-x)$ is the OPPOSITE SIGNS of the original.

If neither: Neither is true.

(1) $f(x) = 2x^3 + x$

$f(-x) = 2(-x)^3 + (-x)$

$f(-x) = -2x^3 - x$

$f(-x) = -f(x)$

So $f(x)$ is ODD.

(2) $f(x) = -3x^2 - x$

$f(-x) = -3(-x)^2 - (-x)$

$f(-x) = -3x^2 + x$

$f(-x)$ is not $f(x)$ nor $-f(x)$.

So $f(x)$ is NEITHER.

(3) $f(x) = \frac{1}{2}x^4 - 2x^2 + 4$

$f(-x) = \frac{1}{2}(-x)^4 - 2(-x)^2 + 4$

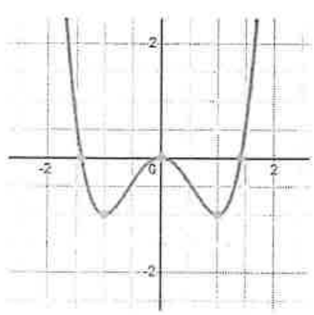
$f(-x) = \frac{1}{2}x^4 - 2x^2 + 4$

$f(-x) = f(x)$

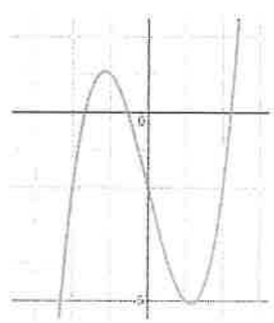
So $f(x)$ is EVEN.

Classify each of the following as even, odd, or neither. Explain/justify your answer.

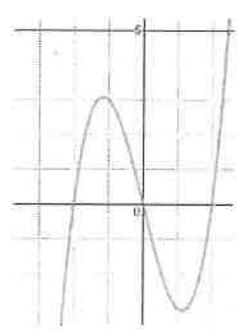
1.



2.



3.



4. $f(x) = 3x^3 - 7x$

5. $g(x) = 3x^8 + 4x^2 - x$

Find the inverse of:

a) $f(x) = 2x + 5$

$x = 2y + 5$

$x - 5 = 2y$

$y = \frac{x-5}{2}$

$y = \frac{1}{2}x - \frac{5}{2}$

$f^{-1}(x) = \frac{1}{2}x - \frac{5}{2}$

b) $g(x) = \sqrt[3]{x-3}$

$x = \sqrt[3]{y-3}$ Switch x and y.

$x^3 = \left(\sqrt[3]{y-3}\right)^3$ Solve for y.

$x^2 = y - 3$

$y = x^2 + 3$

$g^{-1}(x) = x^2 + 3$ Replace y using inverse notation.

Find the inverse of each of the following.

1. $f(x) = 2x + 12$

2. $g(x) = \sqrt{x+4}$

3. $k(x) = 3(x-2) + 4$

4. $m(x) = 2\sqrt{x} - 1$

If you're given the inverse, you can also use this solving strategy to find the original function.

Skill #19: Polynomial Graphs

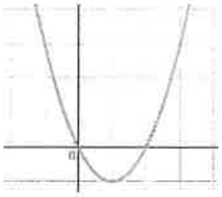
- Sketching the graph of odd and even degree polynomials. Know your end behaviors!

Note: Even and odd degree polynomials are different from even/odd FUNCTIONS. (See Skill #18.)

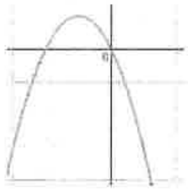
EVEN DEGREE: End behavior is either $\nwarrow \quad \nearrow$ if leading coefficient (L.C.) is positive or $\swarrow \quad \searrow$ if leading coefficient (L.C.) is negative.

<u>Quadratic Graphs (x^2)</u>
Degree: 2
Max # of Roots: 2

Positive L.C.:

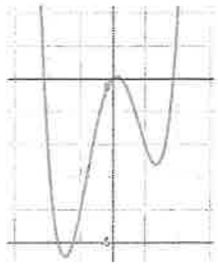


Negative L.C.

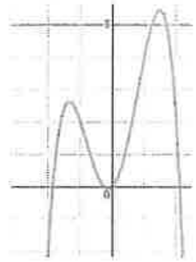


<u>Quartic Graphs (x^4)</u>
Degree: 4
Max # of Roots: 4

Positive L.C.:



Negative L.C.:



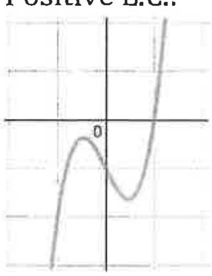
<u>Positive L.C. End Behavior:</u> As $x \to \infty, f(x) \to \infty$. As $x \to -\infty, f(x) \to \infty$.

<u>Negative L.C. End Behavior:</u> As $x \to \infty, f(x) \to -\infty$. As $x \to -\infty, f(x) \to -\infty$.

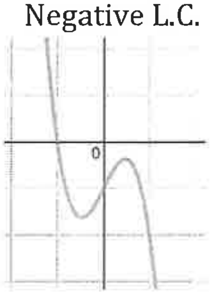
ODD DEGREE: End behavior is either $\swarrow \quad \nearrow$ if leading coefficient (L.C.) is positive or $\nwarrow \quad \searrow$ if leading coefficient (L.C.) is negative.

<u>Cubic Graphs (x^3)</u>
Degree: 3
Max # of Roots: 3

Positive L.C.:

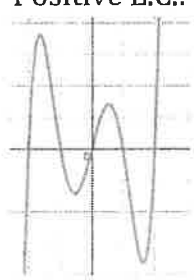


Negative L.C.

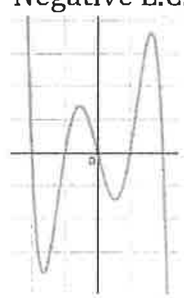


<u>Quintic Graphs (x^5)</u>
Degree: 5
Max # of Roots: 5

Positive L.C.:



Negative L.C.:

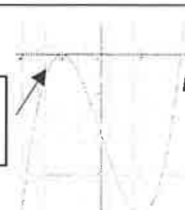


<u>Positive L.C. End Behavior:</u> As $x \to \infty, f(x) \to \infty$. As $x \to -\infty, f(x) \to -\infty$.

<u>Negative L.C. End Behavior:</u> As $x \to \infty, f(x) \to -\infty$. As $x \to -\infty, f(x) \to \infty$.

Multiplicity of A Root: How many times a root "occurs."
-If the graph crosses the x-axis at that point, the multiplicity is 1.
-If the graph bounces on the x-axis at that point, the multiplicity is 2.

Bounces at $x = -1$
Multiplicity = 2



Crosses at $x = 2$
Multiplicity = 1

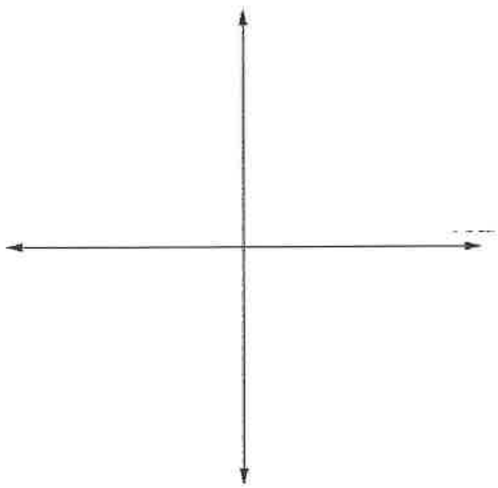
Equation: $(x+1)^2(x-2)^1$ -- The exponent represents the multiplicity.

Fill in the basic information in the chart.

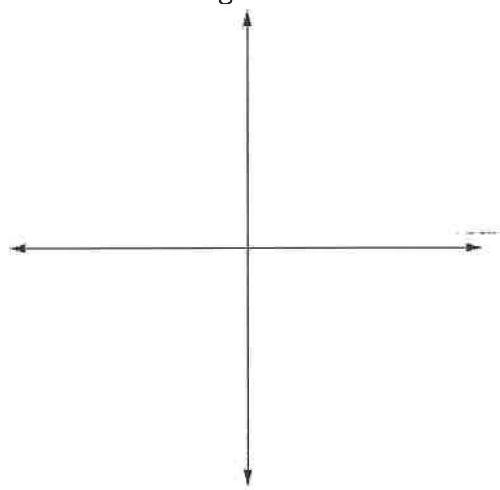
Function Name	**Basic Equation**	**Degree**	**At <u>most</u>, how many different roots?**
Quadratic	$f(x) = x^2$		
Cubic	$f(x) = x^3$		
Quartic	$f(x) = x^4$		
Quintic	$f(x) = x^5$		

Sketch a basic graph given the information <u>and describe the end behavior</u>.

1. Cubic with roots of $-1, 3,$ and 5

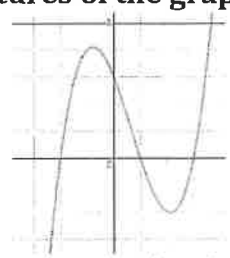


2. Negative quartic function with positive roots a & b and negative roots c & d



Circle the choice that best answers the questions regarding the features of the graph.

3. Which is true regarding the end behavior of the graph?
   (1) As $x \to -\infty, f(x) \to \infty$.
   (2) As $x \to 3, f(x) \to \infty$.
   (3) As $x \to \infty, f(x) \to -\infty$.
   (4) As $x \to -\infty, f(x) \to -\infty$.



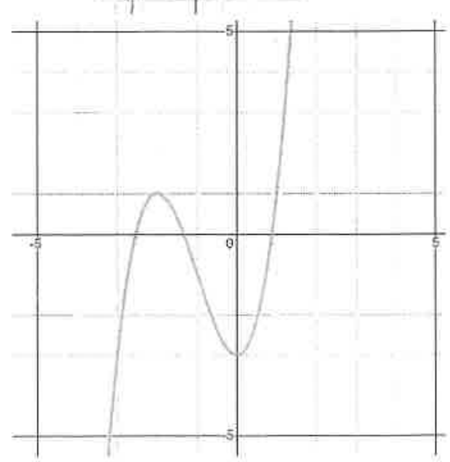
True or False.

4. Using the graph at the right, determine if the following statements are true or false. If false, correct the statement.
   a. It is decreasing on the interval $(-2, 0)$.

   b. It is decreasing on the interval $(0, \infty)$.

   c. It has a relative minimum at the point $(0, -3)$.

   d. It has a relative maximum at the point $(0, -3)$.



<u>Skill #20:</u> Focus/Directrix of Parabola

- The following formula is your friend. Memorize it:

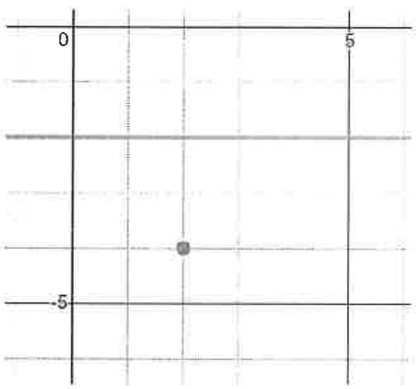
$$y = \pm\frac{1}{4p}(x-h)^2 + k$$

This formula is NOT on the reference sheet. You will have to know it.

A *parabola* is a set of points that is equidistant from a point (*focus*) and a line (*directrix*).

p represents the distance from the vertex to the focus or the directrix.
(h, k) represents the vertex. The x-value of the vertex is always the opposite sign of that in the equation.
Parabolas ALWAYS open toward the focus.
If the parabola opens upward, the coefficient is positive.
If it opens downward, the coefficient is negative.

(1) Write the equation of the parabola equidistant from:



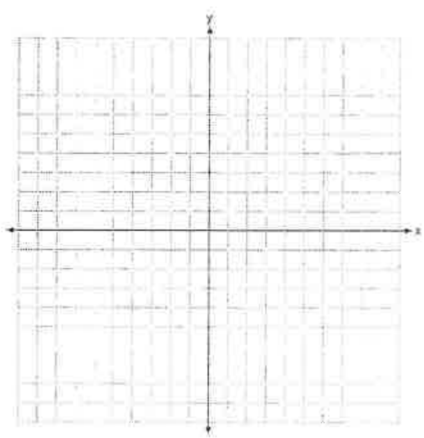
The vertex must be between the focus and directrix, so it must be $(2, -3)$.
The distance from the vertex to the focus is 1, so $p = 1$.
Since the parabola must open toward the focus, this parabola must open downward.

Therefore, the equation must be:

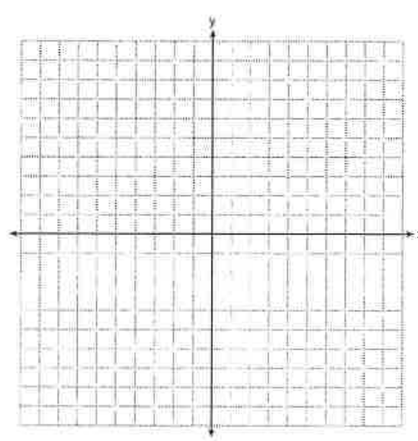
$$y = -\frac{1}{4(1)}(x-2)^2 - 3$$
$$y = -\frac{1}{4}(x-2)^2 - 3$$

Write the equation of the parabola given the following conditions. Use the graph paper if necessary.

1. Focus: $(-2, 4)$
   Directrix: $y = -2$



2. Focus: $(3, 6)$
   Vertex: $(3, 2)$



(2) Find the focus and directrix of $(x+2)^2 = 8(y-3)$.

First, solve for y.

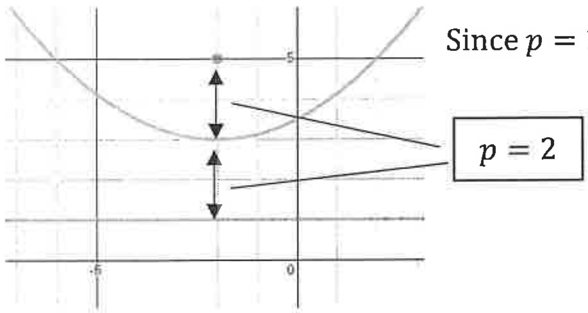
Do not expand $(x+2)^2$.

$$(x+2)^2 = 8y - 24$$
$$8y = (x+2)^2 + 24$$
$$y = \frac{(x+2)^2}{8} + \frac{24}{8}$$
$$y = \frac{1}{8}(x+2)^2 + 3$$

So the vertex is at $(-2, 3)$. Since $\frac{1}{4p} = \frac{1}{8}$, p must equal 2.

The parabola opens upward since the leading coefficient is positive.

Therefore, the parabola must look something like this:



Since $p = 2$, the focus is at $(-2, 5)$ and the directrix is at $y = 1$.

Determine the vertex, focus, and directrix of each parabola. Then sketch a quick graph.

1. $y - 1 = \frac{1}{4}(x-2)^2$

2. $-8(y-2) = (x+3)^2$

3. $4y + x^2 = 0$

4. $y - 1 = (x+5)^2$

Skill #21: Exponential Growth/Decay

- Exponential Growth: Increasing
- Exponential Decay: Decreasing

Generic Exponential: $y = b^x$

Exponential Growth Equations:

Base	Exponent	Example
$b > 1$	*positive*	$y = 3^x$
$0 < b < 1$	*negative*	$y = \left(\frac{1}{3}\right)^{-x}$
		*This simplifies to $y = 3^x$ because the negative exponent will create the reciprocal of $\frac{1}{3}$, which is 3.

Exponential Decay Equations:

Base	Exponent	Example
$0 < b < 1$	*positive*	$y = \left(\frac{1}{2}\right)^x$
$b > 1$	*negative*	$y = 2^{-x}$
		*This simplifies to $y = \left(\frac{1}{2}\right)^x$ because the negative exponent will create the reciprocal of 2 , which is $\frac{1}{2}$.

Complete the chart.

Equation	Growth or Decay?	Asymptote	End Behavior
$f(x) = 4^x$			
$g(x) = 2\left(\frac{1}{2}\right)^{3x}$			
$j(x) = e^{-x}$			
$k(x) = 500(0.75)^{0.77x} + 2$			
$h(t) = 0.88^{-t} - 4$			
$m(x) = \frac{1}{2}(2)^x + 15$			

<u>Skill #22:</u> Exponential Regression (Calculator Skill)

- Combo move: Stat – Edit – Enter data – Stat – Calc – 0: ExpReg

Find the exponential regression equation given the data below. Round to the *nearest hundredth.*

x	y
0	3
1	7
2	10
3	24
4	50
5	95

On you calculator, click STAT – 1:Edit and enter your data into the table.
Click STAT again, arrow over to CALC, and click 0: ExpReg.
Your Xlist should be L_1 and your Ylist should be L_2. Everything else can be left blank. Hit Calculate.

<u>Solution:</u> $y = a * b^x$

$a = 3.046450345$

$b = 1.988034735$

Rounding to the nearest hundredth, we have: $y = 3.05(1.99)^x$

Determine the exponential regression equations and answer the following questions.

1. Bacteria grown in a laboratory after a given number of hours is shown in the table below.

Hour	1	2	3	4	5
Bacteria	1995	2201	2430	2686	2965

Determine the exponential regression equation for this data, rounding values to the *nearest hundredth.*

Assuming the exponential pattern continues, how many bacteria will there be in 8 hours? Round to the *nearest bacterium.*

2. Determine the exponential regression equation, rounding all values to the *nearest thousandth,* given the following data table.

x	y
0	290
1	320
2	400
3	495
4	600
5	700
6	820
7	1000
8	1250
9	1580

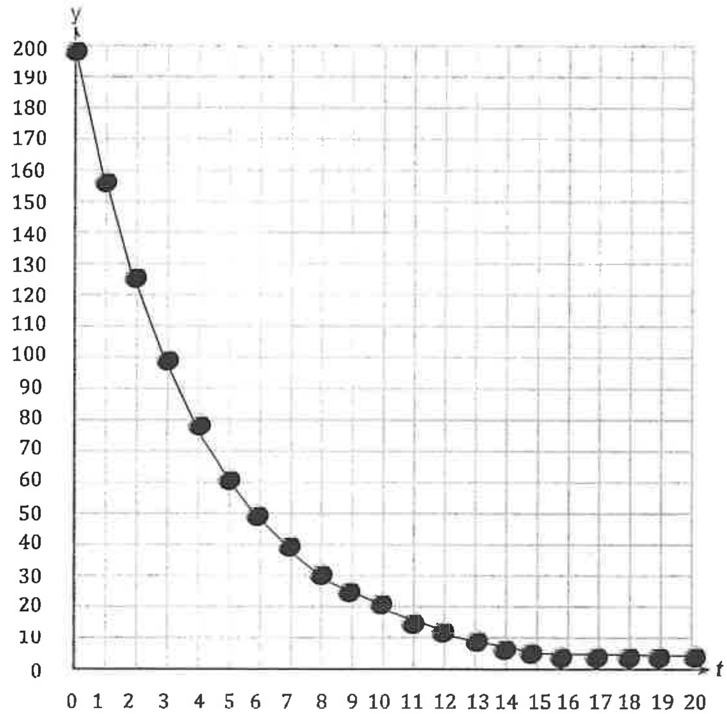
Skill #23: Exponential Graphs

- Make sure you can graph random equations, just in case they ask you to do so. It's not unheard of for your y-values to be decimals.

Given the equation: $A(t) = A_0 e^{-rt}$ where $A(t)$ represents the amount of a drug left in the body after a certain amount of time, A_0 represents the initial amount of a drug in the body, r is the decay rate, and t represents time in hours.
Graph the equation when the initial dosage is 200mg and the decay rate is 0.231.

<u>Solution:</u> This means $A_0 = 200$ and $r = 0.231$. By substitution, we know we are graphing:
$A(t) = 200e^{-0.231t}$

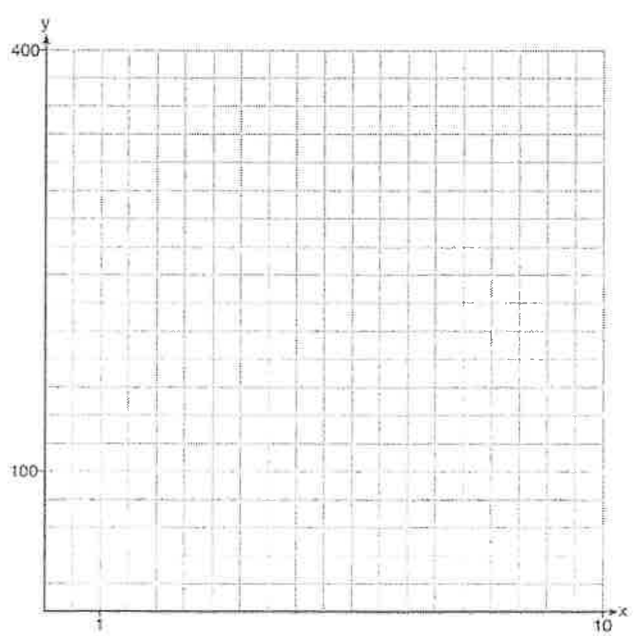
Graph this by typing it into $y_1 = 200e^{-0.231t}$
Look at the table. The highest y-value is 200. Label your axes and plot the points!

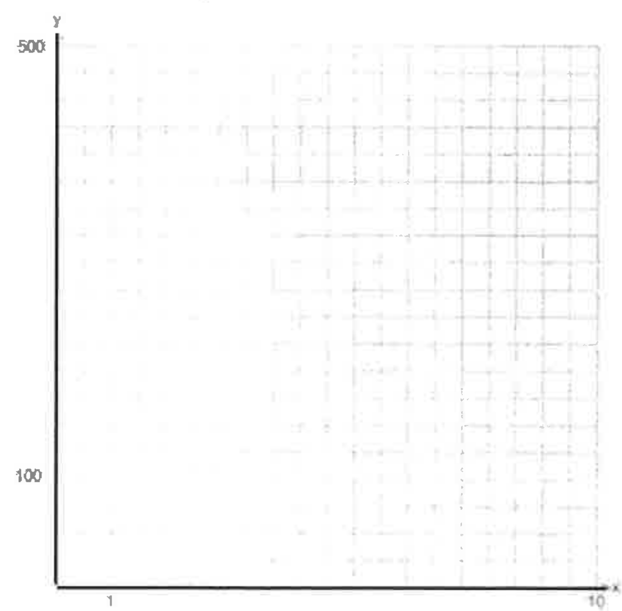


Graph the following on the axes below.

1. $y = 400(0.9)^{3x} - 2$

2. $f(x) = 475(0.75)^{.25x}$





Skill #24: Equivalent Exponential Equations

- Change an equation into an equivalent one by changing the time period.

Given $A = 24(1.075)^t$, where t is time in years, create an equation that will model the approximate:
(1) monthly growth rate (2) weekly growth rate

First, change the exponent to show that there are (1) 12 months in a year, and (2) 52 weeks in a year.

$A = 24\left(1.075^{\frac{1}{12}}\right)^{12t}$ $A = 24\left(1.075^{\frac{1}{52}}\right)^{52t}$

However, to balance our new exponent, we needed to introduce another new exponent to ensure that we are not changing the equation at all.

Since $\frac{1}{12} \cdot 12t = t$ and $\frac{1}{52} \cdot 52t = t$, we can use these as our exponents without changing the value of the equation. We will now simplify the value inside the parentheses using our calculator:

(1) $A = 24(1.006044919)^{12t}$
Since $12t$ represents m months:
$A = 24(1.006044919)^m$

(2) $A = 24(1.00139175)^{52t}$
Since $52t$ represents w weeks:
$A = 24(1.00139175)^{52}$

Complete the following conversions.

1. An antique appreciates according to the equation $f(t) = 500(1.1)^t$, where t is time in years. Determine an equivalent equation that would model the approximate monthly growth rate.

2. The population of a city depreciates according to the equation $y = 25000(0.96)^x$, where x is time in years. Determine an equivalent equation that would model the approximate weekly growth rate.

Multiple Choice. Circle the choice that best answers the question.

3. The amount of visitors to a national park has grown according to the model $P = 3000(1.21)^t$, where t is the time in years. Which of the following equations can model the approximate monthly growth rate in terms of m?

(1) $P = 3000(0.101)^m$ (3) $P = 3000(0.101)^{12m}$
(2) $P = 3000(1.016)^m$ (4) $P = 3000(1.016)^{12m}$

Skill #25: Use and Apply Mortgage Formulas

- Big formulas are no sweat! (Note: It might not always be the formula below.)
- Down Payment: Amount of money paid initially, usually used toward a house or car. The rest of the money needed to buy a house or car usually comes from a loan.

Loren wants to buy a new home for $162,700 near his favorite city. The formula to compute a mortgage payment, M, is $M = P \cdot \frac{r(1+r)^N}{(1+r)^N-1}$ where P is the principal amount of the loan, r is the monthly interest rate, and N is the number of monthly payments. Loren's bank offers a monthly interest rate of 0.205% for a 15-year mortgage.

(1) With no down payment, determine Loren's mortgage payment to the *nearest dollar.*

$N = 15 \cdot 12 = 180$

$$M = 162{,}700 \cdot \frac{0.00205(1+0.00205)^{180}}{(1+0.00205)^{180}-1} = \$1{,}082$$

(2) Algebraically determine and state the down payment, rounded to the *nearest cent*, that Loren needs to make in order for his mortgage payment to be $1000.

$$1000 = P \cdot \frac{0.00205(1+0.00205)^{180}}{(1+0.00205)^{180}-1}$$

$$1000 = 0.0066490789P$$

$$P = \$150{,}396.77 \leftarrow \text{This is your loan amount.}$$

So, your down payment was: $162{,}700 - 150{,}396.77 = \$12{,}303.23$

Using the equation from above, answer the following questions.

1. Find the mortgage payment for a house that costs $280,000, assuming a down payment of $50,000, a mortgage rate of 0.333% for a 30-year mortgage. Round to the *nearest cent.*

2. Determine the down payment needed in order for a mortgage payment to be $1200. Assume the house costs $152,000 at 0.625% monthly interest for 20 years. Round to the *nearest dollar.*

Skill #26: Properties of Logarithms and Their Graphs

- Logarithms and exponentials are inverses!

What is the inverse of $y = 3^x$?

$x = 3^y$ Switch x and y.

$\log_3 x = y$ Solve for y.

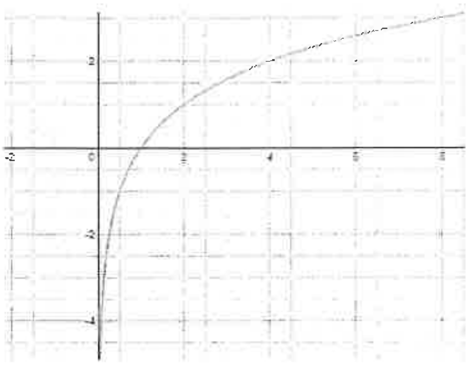
$f^{-1}(x) = \log_3 x$

What is the inverse of $y = \log_2 x$?

$x = \log_2 y$ Switch x and y.

$2^x = y$ Solve for y.

$f^{-1}(x) = 2^x$



Generic Graph of Logarithm:

There is an asymptote at $x = 0$.

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

End Behavior:

As $x \to \infty, f(x) \to \infty$.

As $x \to 0, f(x) \to -\infty$.

(Note: x cannot approach $-\infty$ because of the asymptote.)

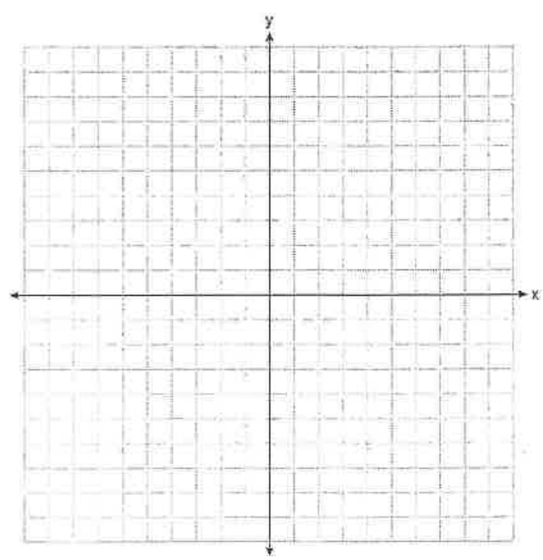
Find the inverse of each of the following.

Function	Inverse
$f(x) = \log_3 x$	
$f(x) = \log_{\frac{1}{3}} x$	
$f(x) = 6^x$	
$f(x) = e^x$	

Answer the following questions regarding graphs of logarithms. Use the graph paper for #2.

1. The graph of $y = \log_2 x$ is translated to the left 1 unit and down 3 units. What is the equation of the translated graph?

2. Graph $y = \log_2(x - 1) + 4$ on the graph paper. Describe the end behavior.



Skill #27: Solving Exponential Equations Using Logarithms

- You must know this conversion: $3^x = 8 \leftrightarrow \log_3 8 = x$

Common Logarithm: $\log_{10} x$ --- often written without the base --- $\log x$
Natural Logarithm: $\log_e x$ --- often written with different notation --- $\ln x$
No matter what type of base you have, they all are approached in the same way!

Solve for x to the *nearest tenth*.

$8(2^{x+7}) + 3 = 37$	First, isolate the exponential. In this case, it is 2^{x+7}.
$8(2^{x+7}) = 34$	Subtract 3 on both sides.
$2^{x+7} = 4.25$	Divide by 8. (Now the exponential is isolated.)
$\log_2 4.25 = x + 7$	Convert into logarithmic form.
$x = \log_2 4.25 - 7$	Subtract 7.
$x = -4.9$	Evaluate using your calculator. (Different log bases can be found by clicking MATH – A:logBASE.)

Note: The answer of $\log_2 4.25 - 7$ is called the *exact* answer since it is not rounded at all.

Solve each of the following equations.

1. Solve to the *nearest tenth:* $4 \cdot 3^n + 15 = 359$

2. Solve $87e^{0.3x} = 5918$ to the *nearest thousandth.*

3. Solve $6 \cdot 16^{7y+2} - 3 = 81$ to the *nearest hundredth.*

Skill #28: Logarithmic Word Problems

- Many times you are asked to apply formulas that are not given. The following are the ones to memorize.

  Compound Interest: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

  Continuous Compound Interest: $A = Pe^{rt}$

  Half-Life: $A = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$

$1200 was placed in a bank account and, after a certain number of years, there was $3000 in the account.

a. How many years have passed if the interest was compounded quarterly at 4.25%?

$$3000 = 1200\left(1 + \frac{0.0425}{4}\right)^{4t}$$

$2.5 = \left(1 + \frac{0.0425}{4}\right)^{4t}$	Divide by 1200.
$2.5 = (1.010625)^{4t}$	Simplify inside parentheses.
$\log_{1.010625} 2.5 = 4t$	Convert to log form.
$t = \frac{\log_{1.010625} 2.5}{4}$	Divide by 4.
$t \approx 21.7$ years	Evaluate.

b. How many years have passed if the interest was compounded continuously at 3%?

$$3000 = 1200e^{0.03t}$$

$2.5 = e^{0.03t}$	Divide by 1200.
$\ln 2.5 = 0.03t$	Convert to log form.
$t = \frac{\ln 2.5}{0.03}$	Divide by 0.03.
$t \approx 30.5$ years	Evaluate.

The half-life of a certain compound is 4 days. If the initial amount of the compound was 100g and now there is 62.4g, how many days have passed?

$$62.4 = 100\left(\frac{1}{2}\right)^{\frac{t}{4}}$$

$0.624 = \left(\frac{1}{2}\right)^{\frac{t}{4}}$	Divide by 100.
$\log_{1/2} 0.624 = \frac{t}{4}$	Convert to log form.
$4\log_{1/2} 0.624 = t$	Multiply by 4.
$t \approx 2.7$ days	Evaluate.

Compound Interest.

1. Nevaeh's parents gave her $2,500 to invest for her 18th birthday. She is considering an investment option that will pay her 3.5% compounded monthly. Algebraically determine, to the *nearest tenth of a year,* how long it would take for this option to double Nevaeh's investment.

2. Joanna deposited $1,000 at 2.8% interest compounded weekly. In how many years, to the *nearest tenth*, will she have $11,000 in the account?

Continuous Compound Interest.

3. After how many years will $100, invested at an annual interest rate of 4% compounded continuously, be worth $450? Round to the *nearest tenth.*

4. In 2000, there was an influx of a new species of insect in a local park. This new insect population is growing continuously at a rate of 8% per year. If the park initially had 50 new insects, in what year will there be three times that number?

Half-Life.

5. Sodium iodide-131, used to treat certain medical conditions, has a half-life of 1.8 hours. A patient took a 150 mcg dose of sodium iodide-131. Determine, to the *nearest tenth of an hour*, how long it will take before the amount in her body will reduce to 30 mcg.

6. A given substance has a half-life of 6,000 years. After t years, one-fifth of the original sample remains radioactive. Find t to the *nearest thousand years.*

Skill #29: Recursive Formulas for Sequences

- These are formulas that are based on the previous term. They are usually written in terms of a_{n-1}. You must include the first term, a_1!

Arithmetic:

Write a recursive formula for: $4, -1, -6, -11, \ldots$

Solution: $a_1 = 4$ You must include the first term!

$a_n = a_{n-1} - 5$

Take the **previous term** and subtract 5.

Geometric:

Write a recursive formula for: $3, 4.5, 6.75, 10.125, \ldots$

Solution: $a_1 = 3$ You must include the first term!

$a_n = 1.5a_{n-1}$

Multiply the **previous term** by 1.5.

Neither Arithmetic Nor Geometric:

Write a recursive formula for:

$2, 5, 11, 23, \ldots$

Solution: $a_1 = 2$

You must include the first term!

$a_n = 2a_{n-1} + 1$

Take the **previous term** and multiply it by 2, then add 1.

Identify if the sequence is arithmetic, geometric, or neither. Write a recursive formula for each of the following sequences.

1. $-4, -6, -8, -10, \ldots$
2. $19, 13, 7, 1, \ldots$
3. $25, 75, 225, \ldots$
4. $3, 9, 27, \ldots$
5. $-1, -4, -13, \ldots$
6. $1, 5, 17, 53, \ldots$

Application.

7. At her job, Monica earns \$35,000 the first year and receives a raise of \$1,500 each successive year. Write a recursive formula that will model her salary.

8. Find the first 5 terms of the sequence: $a_1 = 2, a_n = 2a_{n-1} + 5$

Skill #30: Explicit Formulas for Sequences

- These formulas are <u>given</u> to you on the Regents.
  Arithmetic: $a_n = a_1 + (n-1)d$ Geometric: $a_n = a_1 r^{n-1}$

Arithmetic sequences are those in which each successive term is created by adding or subtracting a certain amount. This amount is called the *common difference, d.*

Write an explicit formula for the sequence: $4, -1, -6, -11, \ldots$

Solution: Since each term is created by subtracting 5, then $d = -5$.
Substituting into the formula, we have:

$a_n = 4 + (n-1)(-5)$
$a_n = 4 - 5n + 5$ Distribute the -5.
$a_n = 9 - 5n$ Simplify.

Geometric sequences are those in which each successive term is created by multiplying by a certain amount. This amount is called the *common ratio, r.*

Write an explicit formula for the sequence: $3, 4.5, 6.75, 10.125, \ldots$

Solution: Since each term is created by multiplying by 1.5, then $r = 1.5$.
Substituting into the formula, we have:

$a_n = 3(1.5)^{n-1}$

There are some ways to simplify this, but, for the most part, you don't see it simplified often.

Identify if the sequence is arithmetic or geometric. Write an explicit formula for each of the following sequences. If the sequence is arithmetic, be sure to simplify your formula completely.

1. $-2, 4, 10, 16, \ldots$
2. $4, 10, 25, 62.5, \ldots$
3. $14, 3, -8, -19, \ldots$
4. $63, 21, 7, \frac{7}{3}, \ldots$

Application.

5. Monica deposited 1 cent into a bank account on the first day of the month. She then deposited 3 cents on the second day, 9 cents on the third day, and 27 cents on the fourth. Assuming the pattern continues, write an explicit formula to represent this scenario.

6. Given an arithmetic sequence with $a_3 = -9$ and $a_7 = 7$, determine the common difference.

7. Given a geometric sequence with $a_2 = 6$ and $a_6 = 1536$, determine two values for the common ratio.

<u>Skill #31:</u> Summation Formulas for Sequences (Series)

- Quickest way to take a sum is to use the summation (Greek sigma) Σ button on your calculator. For geometric series, there is also this formula <u>given</u> to you on the reference sheet: $S_n = \frac{a_1 - a_1 r^n}{1-r}$

Find the sum of the first 10 terms of the sequence: $4, -1, -6, -11, \ldots$

Since this sequence is <u>arithmetic</u>, the sigma notation will be the only method we will use to take the sum.

First, find the term formula:

$$a_n = a_1 + (n-1)d$$
$$a_n = 4 + (n-1)(-5)$$
$$a_n = 9 - 5n$$

Then, use the summation symbol around the term formula:

last term

$$S_{10} = \sum_{n=1}^{10} (9 - 5n) = -185$$

first term

Note: The sigma symbol can be found under MATH --- 0:summation

Find the sum of the first 12 terms of the sequence: $2, -6, 18, -54, \ldots$

Since this sequence is <u>geometric</u>, we can use two different methods to solve.

<u>Method 1:</u> sigma notation

First, find the term formula:

$$a_n = a_1 r^{n-1}$$
$$a_n = 2(-3)^{n-1}$$

Then, use the summation symbol around the term formula:

last term

$$S_{12} = \sum_{n=1}^{12} (2(-3)^{n-1}) = -265720$$

first term

<u>Method 2:</u> formula from reference sheet

$$S_n = \frac{a_1 - a_1 r^n}{1 - r}$$

$$S_{12} = \frac{2 - 2(-3)^{12}}{1 - (-3)} = -265720$$

Determine whether your sequence is arithmetic or geometric. Then find the sum using an appropriate method.

1. Find the summation of $6, 1, -4, -9, \ldots$ through 11 terms.

2. Find the summation of $50, 12.5, 3.125, \ldots$ through 10 terms.

3. Find the summation of $2, -6, 18, \ldots$ through 12 terms.

4. Given the recursive formula: $a_1 = 3, a_n = a_{n-1} + 3$, find the sum of the first 23 terms of this sequence.

5. Given the recursive formula: $a_1 = -2, a_n = 1.5a_{n-1}$, find the sum of the first 8 terms of this sequence.

6. In a geometric sequence, $a_2 = 10$ and $a_4 = 250$. Find the sum of the first 9 terms of this sequence, assuming all terms in the sequence are positive.

7. In an arithmetic sequence, $a_2 = -1$ and $a_5 = 22$. Find the sum of the first 12 terms of this sequence.

Skill #32: Sequences/Series Word Problems

- Apply the aforementioned formulas to word problems.

Jessica's starting salary is \$44,000. Each year, she is expected to earn 3% more than the year before. Find Jessica's total earnings after 8 years.

Solution: Since Jessica is earning 3% more each year, her salaries will create a geometric sequence. Also, $r \neq 0.03$, rather $r = 1.03$ since she is taking 100% of her salary and adding 3% on every year.

Since we need to find the total, we will need to find the sum. I will use the sum formula given on the regents reference sheet. (You could also use sigma Σ notation.)

$$S_n = \frac{a_1 - a_1 r^n}{1 - r}$$

$$S_8 = \frac{44000 - 44000(1.03)^8}{1 - 1.03} = \$391{,}262.79$$

Using the arithmetic and geometric sequence term formulas and summation formulas, answer the following questions.

1. You decide to sign up for a couch-to-10k program. The program suggests jogging for 12 minutes daily during the first week. Each week thereafter, your daily jogging time will increase by 6 minutes. How much daily jogging time will happen during the 10th week?

2. Barry has a \$25,000 salary in the first year of his career. Each year, he gets a 4% raise.
   a. Write an explicit and recursive formula that models Barry's salary.

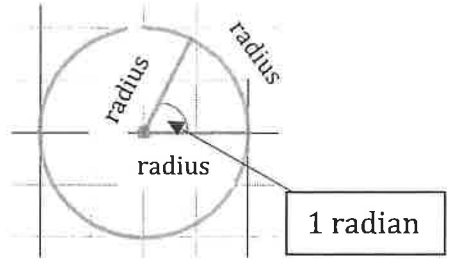
   b. How much total money does Barry earn in the first 11 years of his career?

3. A ball is dropped from a height of 18 feet. The ball retains 60% of its previous height with each bounce. How high, to the *nearest tenth of a foot,* does the ball bounce on the 5th bounce?

Skill #33: Understanding Trigonometry

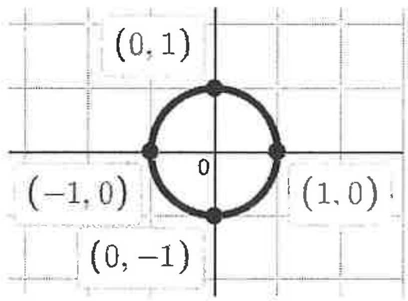
- How does the unit circle work? What are radians? That kind of thing.

An angle measure of **1 radian** is approximately 57.3°. It is the central angle created by wrapping one radius length around the outside of the circle.
- One full circle, or 360°, is equal to 2π radians.
- One half circle, or 180°, is equal to π radians.
To convert from degrees to radians, multiply by $\frac{\pi}{180}$! We generally leave our answer in terms of π, but you may see decimal form as well.



$$150° \cdot \frac{\pi \; radians}{180°} = \frac{150\pi}{180} \; radians = \frac{5\pi}{6} \; radians \approx 2.618 \; radians$$

The unit circle is the circle with radius length of 1 unit.
The x-values on the unit circle represent cosine values.
The y-values on the unit circle represent sine values.
Tangent is represented by $\frac{y}{x} = \frac{sine}{cosine}$.
Quadrant I: sine, cosine, & tangent are positive
Quadrant II: sine is positive, cosine & tangent are negative
Quadrant III: tangent is positive, sine & cosine are negative
Quadrant IV: cosine is positive, sine & tangent are negative



Convert the following degrees to radians. Leave your answer in terms of π.

1. 150°
2. 60°
3. 215°

Convert the following radians to degrees. Remember, π is equal to 180°.

4. $\frac{2\pi}{3}$
5. $\frac{5\pi}{4}$
6. $\frac{\pi}{2}$

Using your knowledge of trigonometry, answer the following questions.

7. Explain why tangent in the third quadrant is always positive.

8. What quadrant are you in if:
   a. $\tan\theta > 0$ and $\cos\theta > 0$
   b. $\cos\theta < 0$ and $\sin\theta < 0$

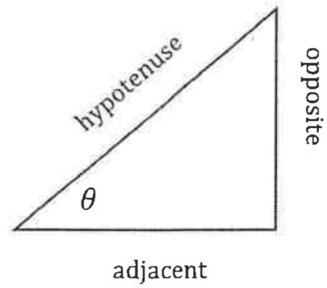
Skill #34: Trigonometric Algebra

- Finding values for the six trig ratios. Also, using $\sin^2\theta + \cos^2\theta = 1$ to solve.

SINE: $\sin\theta = \dfrac{opposite}{hypotenuse}$ COSECANT: $\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{hypotenuse}{opposite}$

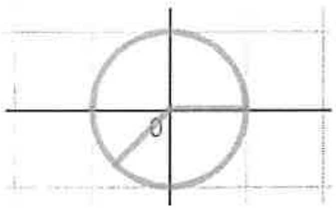
COSINE: $\cos\theta = \dfrac{adjacent}{hypotenuse}$ SECANT: $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{hypotenuse}{adjacent}$

TANGENT: $\tan\theta = \dfrac{opposite}{adjacent}$ COTANGENT: $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{adjacent}{opposite}$

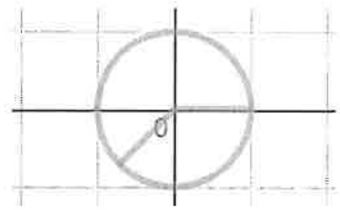


Using the unit circle and the fact that (x, y) on the unit circle represents $(\cos\theta, \sin\theta)$, explain why:

1. $\sec\theta = \frac{1}{x}$



2. $\csc\theta = \frac{1}{y}$



If $\sin^2\theta + \cos^2\theta = 1$, $\sin\theta = -0.5$, and θ is in Quadrant 3, find $\tan\theta$ to the *nearest hundredth.*

$$(-0.5)^2 + \cos^2\theta = 1$$
$$0.25 + \cos^2\theta = 1$$
$$\cos^2\theta = 0.75$$
$$\sqrt{\cos^2\theta} = \sqrt{0.75}$$
$$\cos\theta = \pm 0.8660254038$$

In Quadrant 3, cosine is negative so: $\cos\theta = -0.8660254038$

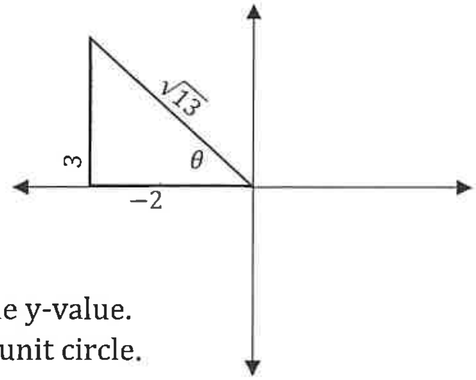
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-0.5}{-0.8660254038} = 0.5773502692 = 0.58$$

Using $sin^2\theta + cos^2\theta = 1$, find each of the following.

3. Given $\cos\theta = -\frac{\sqrt{2}}{5}$ and θ is in Quadrant 2, find $\sin\theta$ in radical form.

4. Given $\sin\theta = -0.33$ and θ is in Quadrant 4, find $\tan\theta$ to the *nearest thousandth.*

An angle, θ, is in standard position and its terminal side passes through the point $(-2, 3)$. Find the exact values of the other trig ratios.
*Since the point is in Q2, we will draw a right triangle using these values for the sides.**
Using Pythagorean Theorem, we can solve for the hypotenuse.
$3^2 + (-2)^2 = c^2 \Rightarrow 9 + 4 = c^2 \Rightarrow c^2 = 13 \Rightarrow c = \sqrt{13}$



Note: Since the hypotenuse is $\sqrt{13}$ and not 1, this is NOT the unit circle!
In other words, you can't just say that cosine is the x-value and sine is the y-value.
That mentality only works if you are told that you are working with the unit circle.

So $\sin\theta = \frac{3}{\sqrt{13}}$ $\csc\theta = \frac{\sqrt{13}}{3}$ $\tan\theta = -\frac{3}{2}$

$\cos\theta = -\frac{2}{\sqrt{13}}$ $\sec\theta = -\frac{\sqrt{13}}{3}$ $\cot\theta = -\frac{2}{3}$

Find the measures of the six trigonometric ratios given the following information.

5. An angle, θ, is in standard position and its terminal side passes through the point $(4, -5)$.

6. An angle, θ, is in standard position and its terminal side passes through the point $(-1, -2)$.

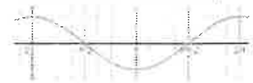
Skill #35: Analyzing Trigonometric Graphs

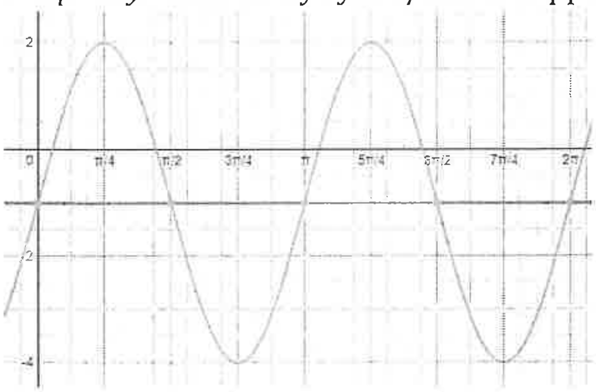
- You should be able to identify the amplitude, period, frequency, and midline from a graph, as well as be able to graph one yourself!

<u>General Features of Sine and Cosine (Sinusoidal) Graphs:</u>
Midline: The horizontal line that cuts the graph in half.
Amplitude: The vertical distance between the midline and the maximum or minimum of the graph.
Period: The time it takes the graph to complete one cycle/wave.
Frequency: How many cycles/waves happen in 2π units.

Generic Sine:

Generic Cosine:





Midline: $y = -1$ *Amplitude:* 3
Period: π *Frequency:* 2

Remember, $period \cdot frequency = 2\pi$!

Equation: $y = 3\sin(2x) - 1$

amp freq midline

<u>Note:</u> The amplitude is never negative because it is a distance. However, the leading coefficient of the equation may be negative. This indicates that the graph was reflected over the x-axis.

Identify the following features from each graph. Then write the equation.

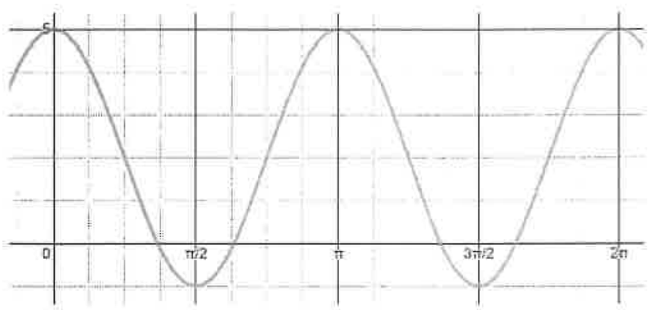
1. Sine or Cosine? ______________

   Midline: ______________

   Amplitude: ______________

   Frequency: ______________

   Period: ______________



Equation: ______________________

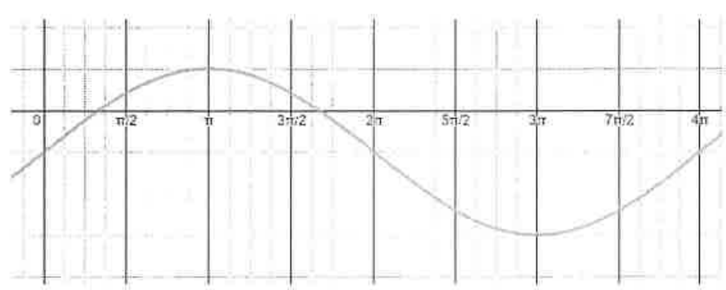
2. Sine or Cosine? ______________

   Midline: ______________

   Amplitude: ______________

   Frequency: ______________

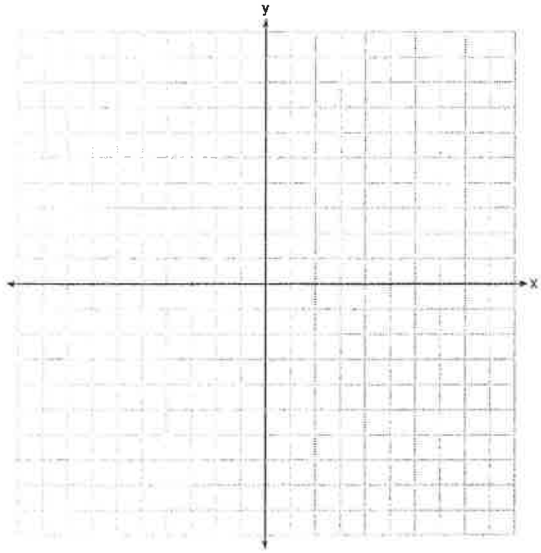
   Period: ______________



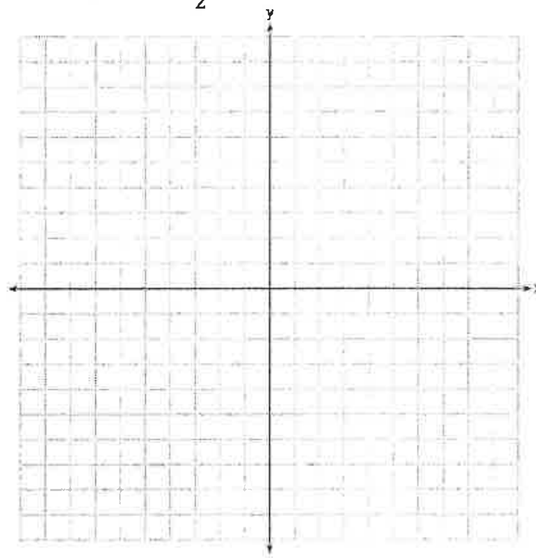
Equation: ______________________

Graph at least one cycle of the function with the given features.

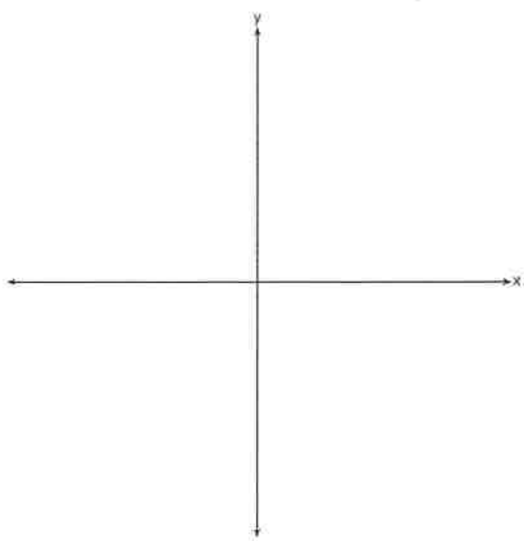
3. Cosine function with
   amplitude of 2
   midline $y = 2$
   period π



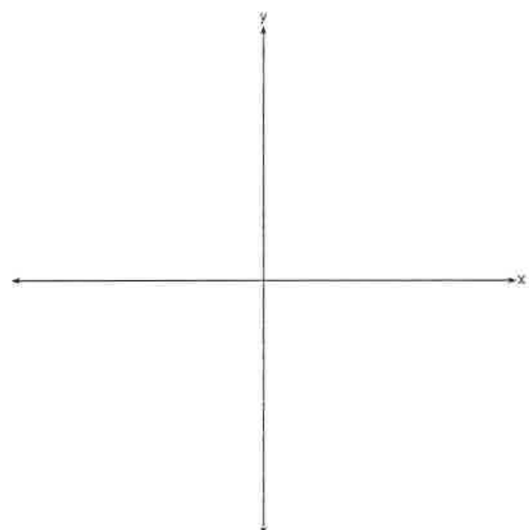
4. Sine function with
   amplitude of 3
   midline $y = -1$
   period $\frac{\pi}{2}$



5. Sine curve with
   midline of $y = \frac{5}{2}$
   amplitude of 3
   period 12



6. Cosine curve with
   midline of $y = -\frac{1}{2}$
   amplitude of 2
   period $\frac{\pi}{4}$



Skill #36: Analyzing Trigonometric Equations

- You should be able to identify the amplitude, period, frequency, and midline from an equation, as well as be able to write one yourself!

$$y = a\sin(bx + c) + d$$

where a is the amplitude, b is the frequency, c is the phase (left/right) shift → + if left, - if right, and d is the vertical shift (midline) → + if up, - if down.

In the equation $y = 2\sin(4x - \pi) + 1$, the following properties exist:

$amplitude = 2$

$frequency = 4$

$period = \frac{\pi}{2}$ (since $f \cdot p = 2\pi \Rightarrow 4p = 2\pi \Rightarrow p = \frac{2\pi}{4} = \frac{\pi}{2}$)

$phase\ shift = right\ \pi$

$vertical\ shift = up\ 1$

$midline\ is\ at\ y = 1$

Given the equations below, determine the following properties.

1. $f(x) = 3\sin(4x) + 2$ Amp: ______ Freq: ______ Per: ______

   Midline: ______ Phase Shift: ____________ Max: ______ Min: ______

2. $f(x) = -4\sin(x - \pi)$ Amp: ______ Freq: ______ Per: ______

   Midline: ______ Phase Shift: ____________ Max: ______ Min: ______

3. $f(x) = \cos(\pi x) - 1$ Amp: ______ Freq: ______ Per: ______

   Midline: ______ Phase Shift: ____________ Max: ______ Min: ______

4. $f(x) = \frac{1}{2}\cos(2x) + 3$ Amp: ______ Freq: ______ Per: ______

   Midline: ______ Phase Shift: ____________ Max: ______ Min: ______

5. $f(x) = 2\cos(x + 1) - 1$ Amp: ______ Freq: ______ Per: ______

   Midline: ______ Phase Shift: ____________ Max: ______ Min: ______

6. $f(x) = 2\sin\left(\frac{\pi}{2}x\right)$ Amp: ______ Freq: ______ Per: ______

   Midline: ______ Phase Shift: ____________ Max: ______ Min: ______

Skill #37: Applications of Trigonometry

- Make sure your calculator is in RADIAN mode at all times!

The amount of energy used at a hangar at an air station, $e(x)$, in kilowatt-hours, is given by $e(x) = 23.91\sin(0.5325\,t - 2.6918) + 82.4484$, where t is the time in months (January: t = 1).

a. State, to the *nearest tenth*, the average monthly rate of energy chance between March and July.

   To find the energy value for March, plug $t = 3$ into your equation, or look at the table on your calculator. In either case, we find that $e(3) = 61.2017971$. Doing the same for July, we have that $e(7) = 103.0162486$.
   Finding the rate of change is the same as finding the slope:
   $m = \frac{103.0162486 - 61.2017971}{7-3} = 10.45361288 = 10.5$ kilowatt-hours per month

b. Determine the maximum and minimum number of kilowatt-hours used in the year.

   Maximum = midline + amplitude = $82.4484 + 23.91 = 106.3584$ kilowatt-hours
   Minimum = midline – amplitude = $82.4484 - 23.91 = 58.5384$ kilowatt-hours

Using the properties of trigonometry, answer the following questions.

1. You are riding on a Ferris Wheel whose rotation is modeled by the equation $f(x) = -25\cos(0.209x) + 28$, where $f(x)$ represents the height off the ground in feet and x represents the time in seconds.
   a. Determine the average rate of change between 7.5 seconds and 30 seconds.

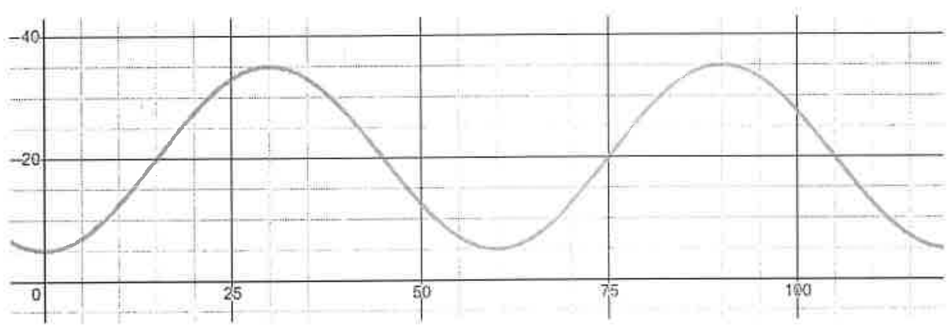
   b. Determine the maximum and minimum height of the rider.

   c. How long does it take for the rider to complete one revolution?

2. The height of the saddle of a horse above the ground on a carousel can be modeled by the equation $f(x) = 12\sin(1.178t) + 42$, where t represents seconds after the ride started.
   a. Determine the period and explain what it means in the context of the problem.

   b. Determine the maximum and minimum height of the saddle above the ground.

3. In the novel *Don Quixote*, Don Quixote gets caught on the tip of one of the sails of a windmill. The graph below represents his height off the ground as a function of time in seconds.



Identify the period of the graph and describe what the period represents in this context.

What was Don Quixote's maximum and minimum height above the ground?

Write an equation that represents this graph.

Skill #38: Probability of Combined Events

- Please be able to know and apply the following formula. It must be memorized. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

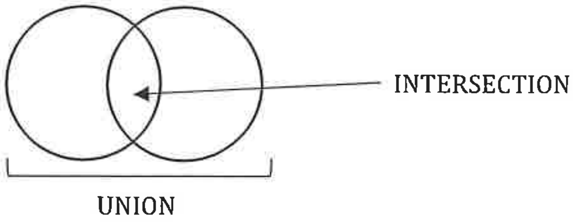
The **union**, ∪, of two sets represents both of the sets combined (including their intersection).

$$A \text{ or } B \text{ or both} = A \cup B$$

The **intersection**, ∩, of two sets represents the overlapping amount between the two sets.

$$\text{both } A \text{ and } B = A \cap B$$

If two sets do not overlap (or do not have an intersection), then they are **disjoint sets/mutually exclusive.**



A high school has an enrollment of 1,702 students. In the school, 577 students are involved in the music program and 701 are involved in the athletics program. If the probability that a student participates in either sports or athletics is $\frac{1077}{1702}$, what is the probability that a student participates in both programs?

$$P(S \cup M) = P(S) + P(M) - P(S \cap M)$$

$$\frac{1077}{1702} = \frac{701}{1702} + \frac{577}{1702} - P(S \cap M)$$

$$\frac{1077}{1702} = \frac{1278}{1702} - P(S \cap M)$$

$$-\frac{201}{1702} = -P(S \cap M)$$

$$P(S \cap M) = \frac{201}{1702}$$

Using a Venn Diagram or the Combined Probability Formula to answer the following questions.

1. Of the current sophomores at a certain high school, 112 like rap music and 150 like pop music. If the sophomore class consists of a population of 327 students and the probability that a sophomore likes rap and pop music is $\frac{67}{327}$, determine the probability that a current sophomore enjoys rap or pop music.